PART B

Unit - V

Numerical Methods - 1

5.1 Introduction

Numerical Methods provides various techniques to find approximate solution to difficult problems using simple operations. Numerical methods are easily adoptable to solve problems using computers as they involve sequential steps.

We discuss (i) numerical solution of algebraic and non algebraic equations.

(ii) numerical solution of a system of linear algebraic equations.

(iii) numerical iterative method to find the largest eigen value and the corresponding eigen vector.

5.2 Numerical Solution of Algebraic and Transcendental Equations

Given an equation $f(x) = 0$, it is generally not possible to find roots x such that $f(x)$ becomes zero exactly. This topic deals with two numerical methods of obtaining approximate real roots of the given equation.

Equations involving algebraic quantities like x, x^2, x^3 etc are called as ***algebraic equations***.

Example : (i) $x^3 - 4x - 9 = 0$ (ii) $x^4 + x^3 = 80$

Equations that involves non algebraic quantities like $e^x, \log x, \sin x, \tan x$ etc. are called as ***transcendental equations.***

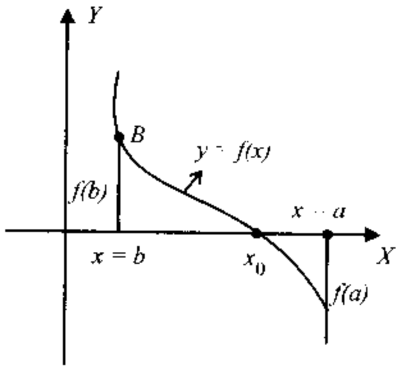
Example : (i) $x e^x - 2 = 0$ (ii) $x \log_e x - 12 = 0$ (iii) $\tan x = 2x$

Numerical methods of finding approximate roots of the given equation is a repetitive type of process known as ***iteration process***. In each step the result of the previous step is used and the process is carried out till we get the result to the desired accuracy. The value obtained in the succeeding step is always better than the value of the preceeding step. All the numerical methods are only approximate techniques for the solution of any problem and computers play a great role in various numerical methods for obtaining the result to the highest degree of accuracy.

If $f(x) = 0$ is a real valued continuous function of the real variable x we have the following fundamental property :

If there exists two values a, b such that $f(x)$ has opposite signs, say $f(a) < 0$, $f(b) > 0$, [Equivalently $f(a) \cdot f(b) < 0$] then there always exist atleast one real root in the interval (a, b).

Geometrically the property means that the graph of $y = f(x)$ intersects the x-axis atleast at one point x_0 that lies between a and b which is a real root of $f(x) = 0$



This property is useful in locating an initial approximation for a real root of $f(x) = 0$

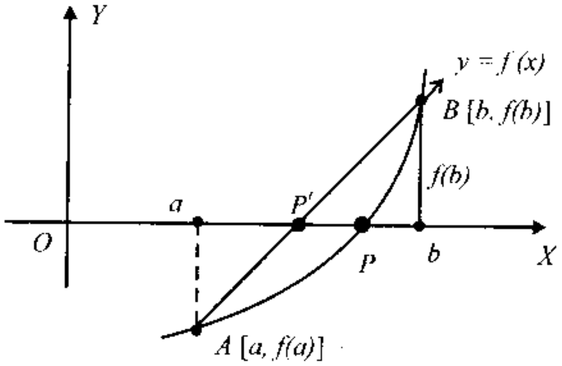
It is **important** to note that ***if the values a, b of x are nearer enough then the numerical iterative methods will give the real root to the desired accuracy quickly.*** In some methods we have to locate an approximate root $x = x_0$ for starting the iterative process. In which case, if the magnitude say $f(a)$ is nearer to zero compared to $f(b)$, we prefer to take $x = x_0 = a$ as the initial approximation.

We discuss two numerical iterative methods.

1. Regula-falsi method 2. Newton-Raphson method

5.21 Regula Falsi Method or Method of False Position

This is a geometrical method for finding an approximate real root of the given equation.



Let the given equation $f(x) = 0$ possess only one real root in the interval (a, b) and let us suppose that $f(a) < 0$, $f(b) > 0$. The graph of $y = f(x)$ in (a, b) crosses the x-axis at only one point P as in the figure.

Solving $f(x) = 0$ is equivalent to, finding x such that $y = 0$, since $y = f(x)$.

That is to find the point where the graph of $y = f(x)$ crosses the x-axis which is nothing but finding the distance OP.

Let $A[a, f(a)]$, $B[b, f(b)]$ be any two points on the curve $y = f(x)$. We know that the equation of the line joining two points $A(x_1, y_1)$, $B(x_2, y_2)$ is given by

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

The equation of the chord AB as in the figure is

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} \qquad \ldots (1)$$

At the point P' where the chord AB crosses the x-axis we must have $y = 0$. If a, b are close enough the point P' tends to P.

Now putting $y = 0$ in (1) we have,

$$\frac{-f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

i.e., $(x - a)[f(b) - f(a)] = -(b - a)f(a)$

i.e., $x[f(b) - f(a)] - af(b) + af(a) = -bf(a) + af(a)$

i.e., $x[f(b) - f(a)] = af(b) - bf(a)$

Thus $$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Remark : *If a, b are close enough we can obtain the approximate root to the desired accuracy quickly. The problems are worked out by finding a and b at a difference of 0.1 to terminate the iterative process quickly.*

WORKED PROBLEMS

1. *Use the regula falsi method to find a real root of the equation $x^3 - 2x - 5 = 0$ correct to three decimal places.*

>> Let $f(x) = x^3 - 2x - 5$

$$f(0) = -5,\ f(1) = -6,\ f(2) = -1 < 0 \quad f(3) = 16 > 0$$

A real root lies in $(2, 3)$

It may be observed that the value of $f(x)$ at $x = 2$ being -1 is nearer to zero compared to $f(3) = 16$ and we expect the root in the neighbourhood of 2. We shall have the interval (a, b) for applying the method such that $b - a$ is small enough.

Now $f(2.1) = (2.1)^3 - 2(2.1) - 5 = +0.061 > 0$

$\therefore$ the root lies in $(2, 2.1)$

I Step : $a = 2, \quad f(a) = f(2) = -1$

$b = 2.1 \quad f(b) = f(2.1) = +0.061$

1st approximation $x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ ... (1)

$$x_1 = \frac{2(0.061) - (2.1)(-1)}{0.061 - (-1)} = 2.0942$$

II Step :

$$f(2.0942) = (2.0942)^3 - 2(2.0942) - 5 = -0.00392 < 0$$

$\therefore$ the root lies in $(2.0942, 2.1)$

Now, $a = 2.0942 \quad f(a) = -0.00392$

$b = 2.1 \quad f(b) = 0.061$

Substituting in the R.H.S of (1) we obtain the second approximation.

$$x_2 = \frac{(2.0942)(0.061) - (2.1)(-0.00392)}{0.061 - (-0.00392)} = 2.09455$$

x_1 and x_2 are close enough. x_2 is a better approximation than x_1.

Thus the required approximate root correct to 3 decimal places is **2.095**

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2. *Compute the real root of* $x \log_{10} x - 1.2 = 0$ *by the method of false position. Carry out three iterations.*

>> Let $f(x) = x \log_{10} x - 1.2$

$$f(1) = -1.2, \quad f(2) = -0.6 < 0, \quad f(3) = 0.23 > 0$$

The real root lies in $(2, 3)$ and from the values of $f(x)$ at $x = 2, 3$ we expect the root in the neighbourhood of 3 and let us find (a, b) for applying the method such that $(b - a)$ is small enough.

$f(2.7) = 2.7 \log_{10} 2.7 - 1.2 = -0.0353$
$f(2.8) = 2.8 \log_{10} 2.8 - 1.2 = +0.052$ } The root lies in $(2.7, 2.8)$

I iteration : $a = 2.7 \qquad f(a) = -0.0353$

$b = 2.8 \qquad f(b) = +0.052$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} \qquad \ldots (1)$$

$$\therefore \quad x_1 = \frac{(2.7)(0.052) - (2.8)(-0.0353)}{0.052 + 0.0353} = 2.7404$$

II iteration : $f(2.7404) = -0.00021 < 0$

$\therefore$ the root lies in $(2.7404, 2.8)$

Now $a = 2.7404 \quad f(a) = -0.00021$

$b = 2.8 \quad f(b) = 0.052$

Substituting in (1) we have

$$x_2 = \frac{(2.7404)(0.052) + (2.8)(0.00021)}{0.052 + 0.00021} = 2.7406$$

III iteration : $f(2.7406) = -0.00004 < 0$

$\therefore$ the root lies in $(2.7406, 2.8)$

Now $a = 2.7406 \quad f(a) = -0.00004$

$b = 2.8 \quad f(b) = 0.052$

Again substituting in (1) we have,

$$x_3 = \frac{(2.7406)(0.052) + (2.8)(0.00004)}{0.052 + 0.00004} = 2.740646$$

Thus the required approximate root correct to four decimal places is **2.7406**

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3. *Use the regula falsi method to find the fourth root of* 12 *correct to three decimal places.*

>> Let $x = \sqrt[4]{12} \quad \therefore \quad x^4 = 12$ or $x^4 - 12 = 0$

Taking $f(x) = x^4 - 12$ we have,

$$f(0) = -12 < 0, \quad f(1) = -11 < 0, \quad f(2) = 4 > 0$$

$\therefore$ a real root of $f(x) = 0$ lies in $(1, 2)$ and will be in the neighbourhood of 2.

Now $f(1.7) = -3.6479, \quad f(1.8) = -1.5024 < 0, \quad f(1.9) = 1.0321 > 0$

$\therefore$ the root in the neighbourhood of 2 lies in $(1.8, 1.9)$

I Step : Let $a = 1.8 \quad f(a) = -1.5024$

$b = 1.9 \quad f(b) = 1.0321$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} \qquad \ldots (1)$$

$$\therefore \quad x_1 = \frac{(1.8)(1.0321) + (1.9)(1.5024)}{1.0321 + 1.5024} = 1.8593$$

II Step : $f(1.8593) = -0.0492 < 0$

$\therefore$ the root lies in $(1.8593, 1.9)$

Now $a = 1.8593 \quad f(a) = -0.0492$

$b = 1.9 \quad f(b) = 1.0321$

Substituting in (1) we obtain $x_2 = 1.8612$

III Step : $f(x_2) = f(1.8612) = -0.00025 < 0$

$\therefore$ the root lies in $(1.8612, 1.9)$

Now $a = 1.8612 \quad f(a) = -0.00025$

$b = 1.9 \quad f(b) = 1.0321$

Substituting in (1) we obtain $x_3 = 1.86121 \approx 1.861$

Thus the required fourth root of 12 correct to 3 decimal places is **1.861**

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4. *Show that a real root of the equation* $\tan x + \tan h\, x = 0$ *lies between* 2 *and* 3. *Then apply the regula falsi method to find the third approximation.*

>> Let $f(x) = \tan x + \tan h\, x$

In radian measure,

$f(2) = \tan 2 + \tan h\, 2 = -1.221 < 0$

$f(3) = \tan 3 + \tan h\, 3 = 0.8525 > 0$

This shows that a real root of $f(x) = 0$ lies in $(2, 3)$.

I approximation : Let $a = 2 \quad f(a) = -1.221$

$b = 3 \quad f(b) = 0.8525$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$\therefore \quad x_1 = \frac{2(0.8525) + 3(1.221)}{0.8525 + 1.221} = 2.59$$

II approximation : $f(x_1) = f(2.59) = 0.3735 > 0$

The root lies in $(2, 2.59)$

Now $a = 2 \qquad f(a) = -1.221$

$b = 2.59 \qquad f(b) = 0.3735$

$$\therefore \qquad x_2 = \frac{2(0.3735) + 2.59(1.221)}{0.3735 + 1.221} = 2.4518$$

III approximation : $f(x_2) = f(2.4518) = 0.1603 > 0$

The root lies in $(2, 2.4518)$

Now $a = 2 \qquad f(a) = -1.221$

$b = 2.4518 \qquad f(b) = 0.1603$

$$\therefore \qquad x_3 = \frac{2(0.1603) + (2.4518)(1.221)}{0.1603 + 1.221} = 2.3994$$

Thus the required third approximation is **2.3994**

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5. *Find the approximate value of the real root of the equation* $x^3 - 3x + 4 = 0$ *using the method of false position (Carry out 3 iterations)*

>> Let $f(x) = x^3 - 3x + 4$

$f(0) = 4 > 0$. $f(x)$ continues to be positive for $x = 1, 2, 3, \ldots$ and hence we shall put $x = -1, -2, -3, \ldots$

$$f(-1) = 6 > 0, \quad f(-2) = 2 > 0, \quad f(-3) = -14 < 0$$

A real root lies in $(-3, -2)$ and we expect the approximation to be in the neighbourhood of -2.

Now, $f(-2.1) = 1.039, \quad f(-2.2) = -0.048$

$\therefore$ the root lies in $(-2.2, -2.1)$

I iteration : $a = -2.2 \qquad f(a) = -0.048$

$b = -2.1 \qquad f(b) = 1.039$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$\therefore \qquad x_1 = \frac{(-2.2)(1.039) - (2.1)(0.048)}{1.039 + 0.048} = -2.196$$

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II iteration : $f(x_1) = f(-2.196) = -0.002 < 0$

The root lies in $(-2.196, -2.1)$

Now $a = -2.196 \quad f(a) = -0.002$

$b = -2.1 \quad f(b) = 1.039$

$$\therefore \quad x_2 = \frac{(-2.196)(1.039) - (2.1)(0.002)}{1.039 + 0.002} = -2.1958$$

III iteration : $f(x_2) = f(-2.1958) = -0.00027 < 0$

The root lies in $(-2.1958, -2.1)$

$a = -2.1958 \quad f(a) = -0.00027$

$b = -2.1 \quad f(b) = 1.039$

$$\therefore \quad x_3 = \frac{(-2.1958)(1.039) - (2.1)(0.00027)}{1.039 + 0.00027} = -2.1958$$

Thus the approximate value of the real root is $-$ **2.1958**

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6. *Find the real root of the equation* $\cos x = 3x - 1$ *correct to three decimals by using regula falsi method.*

>> Let $f(x) = \cos x + 1 - 3x$.

In radians $f(0) = 2 > 0$, $f(1) = -1.46 < 0$

A real root lies in $(0, 1)$ and we expect the root in the neighbourhood of 1.

Consider $f(0.6) = 0.0253 > 0$, $f(0.7) = -0.3352 < 0$

$\therefore$ the root lies in $(0.6, 0.7)$

I iteration : $a = 0.6 \quad f(a) = 0.0253$

$b = 0.7 \quad f(b) = -0.3352$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$\therefore \quad x_1 = 0.607$ *(on substitution and simplification)*

II iteration : $f(x_1) = f(0.607) = 0.00036 > 0$

$\therefore$ the root lies in $(0.607, 0.7)$

Now $a = 0.607 \quad f(a) = 0.00036$

$b = 0.7 \quad f(b) = -0.3352$

$\therefore \quad x_2 = 0.607$ (*on substitution and simplification*).

Hence the real root correct to 3 decimals is **0.607**

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7. *Use the regula falsi method to obtain a root of the equation* $2x - \log_{10} x = 7$ *which lies between* 3.5 *and* 4.

>> Let $f(x) = 2x - \log_{10} x - 7$

$$f(3.5) = 2(3.5) - \log_{10} 3.5 - 7 = -0.5441$$

$$f(4) = 2(4) - \log_{10} 4 - 7 = +0.3979$$

Taking $a = 3.5 \quad f(a) = -0.5441$

$b = 4 \quad f(b) = 0.3979$

We have the first approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$\therefore \quad x_1 = 3.79$ (*on substitution and simplification*)

Also, $f(x_1) = f(3.79) = 0.00136 > 0$

$\therefore$ root lies in $(3.5, 3.79)$

Now $a = 3.5 \quad f(a) = -0.5441$

$b = 3.79 \quad f(b) = 0.00136$

$$\therefore \quad x_2 = \frac{(3.5)(0.00136) + (3.79)(0.5441)}{0.00136 + 0.5441} = 3.7893 \approx 3.79$$

Thus the approximate root correct to two decimal places is **3.79**

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8. *Find the smallest positive root of the equation* $x^2 - \log_e x = 12$ *by Regula - Falsi method.*

>> Let $f(x) = x^2 - \log_e x - 12$

$$f(1) = -11 < 0, f(2) = -8.69 < 0, f(3) = -4.0986 < 0, f(4) = 2.6137 > 0$$

$\therefore$ a real root lies in $(3, 4)$ and will be in the neighbourhood of 4.

Now $f(3.6) = -0.3209$ and $f(3.7) = 0.3817$

I iteration : Let $a = 3.6, \quad f(a) = -0.3209$

$b = 3.7, \quad f(b) = 0.3817$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} \qquad \ldots (1)$$

On substitution we obtain, $x_1 = 3.6457$

II iteration : $f(3.6457) = -0.00242 < 0$

$\therefore$ the root lies in $(3.6457, 3.7)$

Now, $a = 3.6457 \qquad f(a) = -0.00242$

$b = 3.7 \qquad f(b) = 0.3817$

Substituting in (1) we obtain $x_2 = 3.646042$

Thus the required root correct to 3 decimal places is 3.646

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9. *Find the root of the equation* $x e^x - \cos x = 0$ *by the method of false position.*

>> Let $f(x) = x e^x - \cos x$

$f(0) = -1 < 0, \qquad f(1) = 2.178 > 0$

Further we have, $f(0.5) = -0.0532 < 0$, $f(0.6) = +0.2679 > 0$

$\therefore$ the root lies in $(0.5, 0.6)$

I iteration : Let $a = 0.5, \quad f(a) = -0.0532$

$b = 0.6, \quad f(b) = 0.2679$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} \qquad \ldots (1)$$

On substituting we get, $x_1 = 0.5166$

II iteration : $f(0.5166) = -0.0035 < 0$

$\therefore$ the root lies in $(0.5166, 0.6)$

Now, $a = 0.5166, \qquad f(a) = -0.0035$

$b = 0.6, \qquad f(b) = 0.2679$

Substituting in (1) we get $x_2 = 0.5177$

III iteration : $f(0.5177) = -0.00017 < 0$

$\therefore$ the root lies in $(0.5177, 0.6)$

Now, $a = 0.5177, \qquad f(a) = -0.00017$

$b = 0.6, \qquad f(b) = 0.2679$

Substituting in (1) we get $x_3 = 0.5178$

Thus the required root is 0.5178

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10. *Find the negative root of the equation* $x^3 - 4x + 9 = 0$ *from the regula - falsi method.*

>> Let $f(x) = x^3 - 4x + 9$

$$f(0) = 9,\ f(-1) = 12,\ f(-2) = 9,\ f(-3) = -6 < 0$$

Negative root lies in $(-3, -2)$ and will be in the neighbourhood of -3.

Now $f(-2.8) = -1.752,\ f(-2.7) = +0.117$

$\therefore$ the root lies in $(-2.8, -2.7)$

I iteration : Let $a = -2.8,\quad f(a) = -1.752$

$b = -2.7,\quad f(b) = +0.117$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} \qquad \ldots (1)$$

On substitution we get $x_1 = -2.7063$

II iteration : $f(-2.7063) = 0.0041 > 0$

Now, let $a = -2.8,\quad f(a) = -1.752$

$b = -2.7063,\quad f(b) = 0.0041$

Substituting in (1) we get $x_2 = -2.7065$

Thus the required negative root is – 2.7065

5.22 Newton - Raphson method

In this method we locate an approximate real root of the given equation and improve its accuracy by an iterative process.

Let $f(x) = 0$ be the given equation and let x_0 be an approximate root of the equation. If h is a small correction applied to the root then $x_0 + h$ is the exact root and we try to find h such that $f(x_0 + h) = 0$

Using Taylor's expansion of $f(x_0 + h)$ we have,

$$f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

Since h is a small quantity, $h^2, h^3, \ldots$ being still smaller can be neglected.

Hence we have

$$f(x_0) + h f'(x_0) = 0 \quad \text{or} \quad h = -\frac{f(x_0)}{f'(x_0)}$$

The first approximation to the root x_0 is given by $x_1 = x_0 + h$. That is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{provided } f'(x_0) \neq 0$$

The second approximation is obtained by replacing x_0 by x_1 in the R.H.S of this expression. That is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \text{and so on.}$$

In general we can write,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is called ***Newton-Raphson iterative formula.***

WORKED PROBLEMS

11. *Use Newton - Raphson method to find a real root of the equation* $x^3 - 2x - 5 = 0$ *correct to three decimal places.*

>> We shall find an interval (a, b) where a real root of the equation lies and then locate the approximate root.

Let $f(x) = x^3 - 2x - 5$

$$f(0) = -5 < 0,\ f(1) = -6 < 0,\ f(2) = -1 < 0,\ f(3) = 16 > 0$$

A real root lies in $(2, 3)$. It will be in the neighbourhood of 2 and let the approximate root $x_0 = 2$.

The first approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}$$

We have $f(x) = x^3 - 2x - 5,\ f'(x) = 3x^2 - 2$

$$\therefore \quad x_1 = 2 - \frac{(-1)}{3(2^2) - 2} = 2 + \frac{1}{10} = 2 \cdot 1$$

Again, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 2.1 - \dfrac{f(2.1)}{f'(2.1)}$

$$x_2 = 2.1 - \frac{[(2.1)^3 - 2(2.1) - 5]}{[3(2.1)^2 - 2]} = 2.0946$$

Similarly $x_3 = 2.0946 - \frac{[(2.0946)^3 - 2(2.0.946) - 5]}{[3(2.0946)^2 - 2]} = 2.0946$

Thus the required approximate root correct to 3 decimal places is **2.095**

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12. *Show that a root of the equation* $x^3 + 5x - 11 = 0$ *lies between* **1** *and* **2**. *Find the root by Newton - Raphson method (carry out 3 iterations)*

>> Let $f(x) = x^3 + 5x - 11$

$f(1) = -5 < 0, \; f(2) = 7 > 0$

$\therefore$ a real root lies in $(1, 2)$ and let $x_0 = 1$

We have $f(x) = x^3 + 5x - 11, \; f'(x) = 3x^2 + 5$

I iteration : $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{(-5)}{[3(1)^2 + 5]} = 1 + \frac{5}{8} = 1.625$$

II iteration : $x_2 = 1.625 - \frac{f(1.625)}{f'(1.625)}$

$$x_2 = 1.625 - \frac{[(1.625)^3 + 5(1.625) - 11]}{[3(1.625)^2 + 5]} = 1.5154$$

III iteration :

$$x_3 = 1.5154 - \frac{[(1.5154)^3 + 5(1.5154) - 11]}{[3(1.5154)^2 + 5]} = 1.5106$$

Thus the required root is **1.5106**

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13. *Use Newton - Raphson method to find a real root of* $x \sin x + \cos x = 0$ *near* $x = \pi$. *Carry out the iterations upto four decimal places of accuracy.*

>> Let $f(x) = x \sin x + \cos x$

$\therefore \quad f'(x) = x \cos x + \sin x - \sin x = x \cos x$

Also $x_0 = \pi$ *(In radian measure)*

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{f(\pi)}{f'(\pi)} = \pi - \frac{(\pi \sin \pi + \cos \pi)}{\pi \cos \pi}$$

$\therefore \quad x_1 = \pi - \frac{1}{\pi} = 2.8233$

Now, $x_2 = 2.8233 - \frac{[\,2.8233 \sin(2.8233) + \cos(2.8233)\,]}{2.8233 \cos(2.8233)}$

$\therefore \quad x_2 = 2.7986$

Now, $x_3 = 2.7986 - \frac{[\,2.7986 \sin(2.7986) + \cos(2.7986)\,]}{2.7986 \cos(2.7986)}$

$\therefore \quad x_3 = 2.7984$

Again replacing 2.7984 in place of 2.7986 as earlier we obtain $x_4 = 2.7984$.

Thus the required real root is **2.7984**

---

14. *Find a real root of the equation* $x^3 + x^2 + 3x + 4 = 0$ *applying Newton - Raphson method. Carryout two iterations.*

>> Let $f(x) = x^3 + x^2 + 3x + 4$

$f(0) = 4, \quad f(-1) = 1, \quad f(-2) = -6 < 0$

$\therefore$ a real root lies in $(-2, -1)$ and let $x_0 = -1$

We also have $f'(x) = 3x^2 + 2x + 3$

I iteration : $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$x_1 = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{(1)}{3-2+3} = -1.25$$

II iteration : $x_2 = -1.25 - \frac{f(-1.25)}{f'(-1.25)}$

$$x_2 = -1.25 - \frac{[\,(-1.25)^3 + (-1.25)^2 + 3(-1.25) + 4\,]}{[\,3(-1.25)^2 + 2(-1.25) + 3\,]}$$

$\therefore \quad x_2 = -1.2229$

Thus the required real root is $\mathbf{-1.2229 \approx -1.223}$

---

15. *Use Newton - Raphson method to find $\sqrt[3]{37}$ correct to 3 decimal places.*

>> Let $x = \sqrt[3]{37}$ $\therefore$ $x^3 = 37$ or $x^3 - 37 = 0$

Taking $f(x) = x^3 - 37$ we have $f(3) = -10 < 0$, $f(4) = 27 > 0$

$\therefore$ a real root lies in $(3, 4)$ and let $x_0 = 3$ be the initial approximation.

Also $f'(x) = 3x^2$

The first approximation $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$x_1 = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{(-10)}{3(3^2)} = 3.3704$$

Now, $x_2 = 3.3704 - \dfrac{f(3.3704)}{f'(3.3704)} = 3.3704 - \dfrac{[(3.3704)^3 - 37]}{3(3.3704)^2}$

$\therefore$ $x_2 = 3.3327$

Now, $x_3 = 3.3327 - \dfrac{[(3.3327)^3 - 37]}{3(3.3327)^2} = 3.3322$

Again replacing 3.3322 in place of 3.3327 as earlier we obtain $x_4 = 3.3322$

Thus the required $\sqrt[3]{37}$ correct to 3 decimal places is **3.332**

---

16. *Find a real root of the equation $xe^x - 2 = 0$ correct to three decimal places using Newton - Raphson method.*

>> Let $f(x) = xe^x - 2$

$f(0) = -2 < 0$, $f(1) = 0.7183 > 0$

$\therefore$ a real root lies in $(0, 1)$ and let $x_0 = 1$

Also $f'(x) = xe^x + e^x = e^x(x+1)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)}$$

$$x_1 = 1 - \frac{0.1783}{e^1(2)} = 0.8679$$

$$x_2 = 0.8679 - \frac{f(0.8679)}{f'(0.8679)} = 0.8679 - \frac{[0.8679\, e^{0.8679} - 2]}{e^{0.8679}(0.8679 + 1)} = 0.8528$$

$$x_3 = 0.8528 - \frac{[\,0.8528\,e^{0.8528} - 2\,]}{e^{0.8528}\,(\,1.8528\,)} = 0.8526$$

Again replacing 0.8526 in place of 0.8528 as earlier, we obtain $x_4 = 0.8526$.

Thus the required real root correct to three decimal places is **0.853**

---

17. *In calculating the height of a vertical column which will buckle under its own weight, it is necessary to solve the equation* $\frac{x^3}{12960} - \frac{x^2}{180} + \frac{x}{6} - 1 = 0$. *Find one of the root near* $x = 8$ *upto 3 decimal places using Newton - Raphson method.*

>> Let $f(x) = \frac{x^3}{12960} - \frac{x^2}{180} + \frac{x}{6} - 1$ and $x_0 = 8$ by data.

$$f'(x) = \frac{x^2}{4320} - \frac{x}{90} + \frac{1}{6}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 8 - \frac{f(8)}{f'(8)}$$

$$= 8 - \frac{\left[\frac{8^3}{12960} - \frac{8^2}{180} + \frac{8}{6} - 1\right]}{\frac{8^2}{4320} - \frac{8}{90} + \frac{1}{6}} = 7.8133$$

Now replacing 7.8133 in place of 8 as above we obtain the second approximation $x_2 = 7.8147$ and on similar lines we obtain $x_3 = 7.8147$.

Thus the required root near $x = 8$ is **7.815**

Use Newton - Raphson method to derive the following and hence compute as mentioned. **[18 to 22]**

18. *An iterative formula to find* $\sqrt{N}$ *and hence find* $\sqrt{12}$

19. *An iterative formula for the reciprocal of the square root of a positive number and hence find* $(15)^{-1/2}$ *correct to four decimal places.*

20. *An interative formula for finding cube root of* N *and hence find cube root of* 10.

21. *An iterative formula for finding the reciprocal of a number and hence find* 1/18.

22. *An iterative formula for finding the* k^{th} *root of a positive number and hence find fourth root of* 22

>> **18.** Let $x = \sqrt{N}$

$\therefore \quad x^2 = N$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$

Newton Raphson iterative formula is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

ie., $$x_{n+1} = x_n - \frac{(x_n^2 - N)}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n} = \frac{x_n^2 + N}{2x_n}$$

Thus $$\boldsymbol{x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]} \quad \ldots (1)$$

This is the required iterative formula for finding $\sqrt{N}$.

To find $\sqrt{12}$ we have to take $N = 12$ and the initial approximation can be located as follows.

$\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{12}$ is closer to $\sqrt{9} = 3$.

Let $x_0 = 3$ and the successive approximations $x_1, x_2, x_3, \ldots$ are extracted from (1) by putting $n = 0, 1, 2, \ldots$

$$x_1 = \frac{1}{2}\left[x_0 + \frac{12}{x_0}\right] = \frac{1}{2}\left[3 + \frac{12}{3}\right] = 3.5$$

$$x_2 = \frac{1}{2}\left[x_1 + \frac{12}{x_1}\right] = \frac{1}{2}\left[3.5 + \frac{12}{3.5}\right] = 3.4643$$

$$x_3 = \frac{1}{2}\left[x_2 + \frac{12}{x_2}\right] = \frac{1}{2}\left[3.4643 + \frac{12}{3.4643}\right] = 3.4641$$

$$x_4 = \frac{1}{2}\left[x_3 + \frac{12}{x_3}\right] = \frac{1}{2}\left[3.4641 + \frac{12}{3.4641}\right] = 3.4641$$

Thus $\boldsymbol{\sqrt{12} = 3.4641}$.

>> **19.** Let $x = \frac{1}{\sqrt{N}} \quad \therefore \quad x^2 = \frac{1}{N}$ or $x^2 - \frac{1}{N} = 0$

Taking $f(x) = x^2 - \frac{1}{N}$, we have $f'(x) = 2x$ and substitution in the Newton-Raphson iterative formula yields

$$x_{n+1} = x_n - \frac{x_n^2 - \frac{1}{N}}{2x_n} = \frac{2x_n^2 - x_n^2 + 1/N}{2x_n} = \frac{x_n^2 + 1/N}{2x_n}$$

Thus $$x_{n+1} = \frac{1}{2}\left[x_n + \frac{1}{Nx_n}\right] \quad \ldots (2)$$

This is the required iterative formula for finding $1/\sqrt{N}$

To find $(15)^{-1/2} = 1/\sqrt{15}$ we have to take $N = 15$ and the initial approximation is located as follows.

Since $1/\sqrt{15}$ is close to $1/\sqrt{16} = 1/4 = 0.25$, we shall take $x_0 = 0.25$ and the successive approximations are obtained from (2) by putting $n = 0, 1, 2, \ldots$.

$$x_1 = \frac{1}{2}\left[x_0 + \frac{1}{15\,x_0}\right] = \frac{1}{2}\left[0.25 + \frac{1}{15 \times 0.25}\right] = 0.2583$$

$$x_2 = \frac{1}{2}\left[x_1 + \frac{1}{15x_1}\right] = \frac{1}{2}\left[0.2583 + \frac{1}{15 \times 0.2583}\right] = 0.2582$$

$$x_3 = \frac{1}{2}\left[x_2 + \frac{1}{15\,x_2}\right] = \frac{1}{2}\left[0.2582 + \frac{1}{15 \times 0.2582}\right] = 0.2582$$

Thus $\mathbf{(15)^{-1/2} = 0.2582}$

>> **20.** Let $x = \sqrt[3]{N} \Rightarrow x^3 = N$ or $x^3 - N = 0$.

Taking $f(x) = x^3 - N$ we have $f'(x) = 3x^2$ and substitution in the Newton - Raphson iterative formula yields

$$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{3x_n^3 - x_n^3 + N}{3x_n^2} = \frac{2x_n^3 + N}{3x_n^2}$$

Thus $$x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right] \quad \ldots (3)$$

This is the required iterative formula for finding $\sqrt[3]{N}$.

To find $\sqrt[3]{10}$, we have to take $N = 10$ and $\sqrt[3]{10}$ is close to $\sqrt[3]{8} = 2$ and hence we shall take $x_0 = 2$.

Also from (3) we obtain

$$x_1 = \frac{1}{3}\left[2x_0 + \frac{10}{x_0^2}\right] = \frac{1}{3}\left[2 \times 2 + \frac{10}{2^2}\right] = 2.1667$$

$$x_2 = \frac{1}{3}\left[2x_1 + \frac{10}{x_1^2}\right] = \frac{1}{3}\left[2 \times 2.1667 + \frac{10}{(2.1667)^2}\right] = 2.1545$$

$$x_3 = \frac{1}{3}\left[2x_2 + \frac{10}{x_2^2}\right] = \frac{1}{3}\left[2 \times 2.1545 + \frac{10}{(2.1545)^2}\right] = 2.1544$$

$$x_4 = \frac{1}{3}\left[2x_3 + \frac{10}{x_3^2}\right] = \frac{1}{3}\left[2 \times 2.1544 + \frac{10}{(2.1544)^2}\right] = 2.1544$$

Thus **$\sqrt[3]{10} = 2.1544$**

>> **21.** Let $x = \frac{1}{N}$ or $\frac{1}{x} = N$ or $\frac{1}{x} - N = 0$

(It may be noted that if we take $f(x) = x - \frac{1}{N}$, $f'(x) = 1$ x_{n+1} *becomes* $1/N$ *itself)*

Taking $f(x) = \frac{1}{x} - N$ we have $f'(x) = -\frac{1}{x^2}$ and substitution in the Newton - Raphson iterative formula will give us

$$x_{n+1} = x_n - \frac{(1/x_n - N)}{-1/x_n^2} = x_n + x_n^2\,(1/x_n - N)$$

i.e., $x_{n+1} = x_n + x_n - Nx_n^2 = 2x_n - Nx_n^2$

Thus $\boldsymbol{x_{n+1} = x_n(2 - Nx_n)}$ . . . (4)

This is the required iterative formula for finding $1/N$

To find $1/18$ we have to take $N = 18$ and we observe that $1/18$ is close to $1/20 = 0.05$ and hence we shall take $x_0 = 0.05$.

Also we obtain from (4),

$$x_1 = x_0\,(2 - 18x_0) = 0.05\,(2 - 18 \times 0.05) = 0.055$$

$$x_2 = x_1\,(2 - 18x_1) = 0.055\,(2 - 18 \times 0.055) = 0.05555$$

$$x_3 = x_2\,(2 - 18x_2) = 0.05555\,(2 - 18 \times 0.05555) = 0.0555555.$$

Thus **$1/18 = 0.05555$**

>> **22.** Let $x = \sqrt[k]{N} = N^{1/k}$ $\therefore$ $x^k = N$ or $x^k - N = 0$

Taking $f(x) = x^k - N$, $f'(x) = k x^{k-1}$.

Substituting in the Newton - Raphson iterative formula we have,

$$x_{n+1} = x_n - \frac{(x_n^k - N)}{k x_n^{k-1}} = \frac{k x_n^k - x_n^k + N}{k x_n^{k-1}}$$

Thus $$\boldsymbol{x_{n+1} = \frac{(k-1)x_n^k + N}{k x_n^{k-1}}} \qquad \ldots (5)$$

This is the required iterative formula for finding $\sqrt[k]{N}$

To find $\sqrt[4]{22}$ we have to take $N = 22$ and $k = 4$.

We obtain from (5),

$$x_{n+1} = \frac{3x_n^4 + 22}{4x_n^3} = \frac{1}{4}\left[3x_n + \frac{22}{x_n^3}\right] \qquad \ldots (6)$$

Further, we know that $\sqrt[4]{16} = 2$ and $\sqrt[4]{81} = 3$.

To find $\sqrt[4]{22}$ we shall take $x_0 = 2$ and we have from (6),

$$x_1 = \frac{1}{4}\left[3x_0 + \frac{22}{x_0^3}\right] = \frac{1}{4}\left[3 \times 2 + \frac{22}{2^3}\right] = 2.1875$$

$$x_2 = \frac{1}{4}\left[3x_1 + \frac{22}{x_1^3}\right] = \frac{1}{4}\left[3 \times 2.1875 + \frac{22}{(2.1875)^3}\right] = 2.1661$$

$$x_3 = \frac{1}{4}\left[3x_2 + \frac{22}{x_2^3}\right] = \frac{1}{4}\left[3 \times 2.1661 + \frac{22}{(2.1661)^3}\right] = 2.1657$$

$$x_4 = \frac{1}{4}\left[3x_3 + \frac{22}{x_3^3}\right] = \frac{1}{4}\left[3 \times 2.1657 + \frac{2}{(2.1657)^3}\right] = 2.1657$$

Thus $\boldsymbol{\sqrt[4]{22} = 2.1657}$

- -

23. *Derive Newton - Raphson iterative formula to find the real root of the equation $x \log_{10} x = 1.2$ and hence find the root correct to five decimal places.*

>> We have Newton - Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \ldots (1)$$

Let $f(x) = x \log_{10} x - 1.2$

$$f'(x) = x \frac{d}{dx}(\log_{10} x) + \log_{10} x \cdot 1$$

$$= x \frac{d}{dx}\left(\log_e x \cdot \log_{10} e\right) + \log_{10} x.$$

But $\log_{10} e = 0.4343$

$$\therefore \quad f'(x) = 0.4343\, x \cdot \frac{1}{x} + \log_{10} x = 0.4343 + \log_{10} x$$

Hence (1) becomes

$$x_{n+1} = x_n - \frac{(x_n \log_{10} x_n - 1.2)}{0.4343 + \log_{10} x_n}$$

Thus $$\mathbf{x_{n+1} = \frac{0.4343\, x_n + 1.2}{0.4343 + \log_{10} x_n}} \qquad \ldots (2)$$

This is the required iterative formula.

Next let us locate an approximate root of the equation $f(x) = 0$, where $f(x) = x \log_{10} x - 1.2$

$$f(1) = -1.2 < 0,\ f(2) = -0.6 < 0,\ f(3) = 0.23 > 0$$

We expect the root in the neighbourhood of 3 and hence let us take $x_0 = 3$ as the initial approximation.

We have from (2) when $n = 0$,

$$x_1 = \frac{0.4343\, x_0 + 1.2}{0.4343 + \log_{10} x_0} = \frac{0.4343 \times 3 + 1.2}{0.4343 + \log_{10} 3} = 2.74615$$

$$x_2 = \frac{0.4343\, x_1 + 1.2}{0.4343 + \log_{10} x_1} = 2.74065$$

$$x_3 = \frac{0.4343\, x_2 + 1.2}{0.4343 + \log_{10} x_2} = 2.74065$$

Thus the required approximate root correct to five decimal places is 2.74065

24. *Using Newton - Raphson method find an approximate root of the equation* $x \log_{10} x = 1.2$ *correct to five decimal places that is near 2.5*

>> [*Referring to the previous problem it may be observed that this problem is worked by deriving an iterative formula*]

Let $f(x) = x \log_{10} x - 1.2$ and $x_0 = 2.5$ by data.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We have $f'(x) = 0.4343 + \log_{10} x$ [*Refer the previous problem*]

Now $x_1 = 2.5 - \dfrac{f(2.5)}{f'(2.5)}$

$$= 2.5 - \frac{[\,2.5 \log_{10} 2.5 - 1.2\,]}{0.4343 + \log_{10} 2.5} = 2.7465$$

Next $x_2 = 2.7465 - \dfrac{[\,2.7465 \log_{10} 2.7465 - 1.2\,]}{0.4343 + \log_{10} 2.7465} = 2.74065$

Similarly on replacing 2.74065 in place of 2.7465 we again obtain 2.74065

Thus the required approximate root is 2.74065

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25. *Using Newton - Raphson method, find the root that lies near* $x = 4.5$ *of the equation* $\tan x = x$ *correct to four decimal places. (Here x is in radians)*

>> Let $f(x) = \tan x - x \quad \therefore \quad f'(x) = \sec^2 x - 1 = \tan^2 x$

Also $x_0 = 4.5$ radians.

I Iteration : $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$= 4.5 - \frac{f(4.5)}{f'(4.5)}$$

$$x_1 = 4.5 - \frac{[\,\tan(4.5) - 4.5\,]}{\tan^2(4.5)} = 4.4936$$

II Iteration : $x_2 = 4.4936 - \dfrac{[\,\tan(4.4936) - 4.4936\,]}{\tan^2(4.4936)}$

$$x_2 = 4.4934$$

Again replacing 4.4934 in place of 4.4936 we obtain $x_3 = 4.4934$

Thus the required real root is 4.4934

EXERCISES

Use Regula-Falsi method to find a real root of the following equations correct to three decimal places.

1. $x^3 - 5x - 7 = 0$

2. $x^5 - x^4 - x^3 - 1 = 0$ in (1.4, 1.5)

3. $x - 3e^{-x} = 0$

4. $x^3 - 9x + 1 = 0$ in (2.5, 3)

5. $x^3 + x^2 + x + 7 = 0$

Use Newton - Raphson method to find a real root of the following equations correct to three decimal places.

6. $x^4 + x^3 - 7x^2 - x + 5 = 0$ in (2, 3)

7. $x^3 - 3x - 5 = 0$

8. $x^2 - \log_e x - 12 = 0$

9. $\log x - \cos x = 0$ near $x = 1.5$

With the help of an appropriate iterative formulae originating from the Newton Raphson method compute the following correct to four decimal places.

10. $(12)^{-1/2}$ 11. $\sqrt{17}$ 12. $1/\sqrt{e}$

13. $\sqrt[3]{22}$ 14. $1/e$ 15. $\sqrt[5]{200}$

ANSWERS

1. 2.747	2. 1.404	3. 1.050
4. 2.741	5. −2.105	6. 2.061
7. 2.279	8. 3.646	9. 1.303
10. 0.2887	11. 4.1231	12. 0.6065
13. 2.8020	14. 0.3679	15. 2.8854

5.3 Iterative methods of solution of a system of algebraic equations

We are familiar with some methods for solving a system of algebraic equations.

We have earlier discussed two methods giving exact solution of a system of algebraic equations [*Vol.1, Unit-VII*]

1. Gauss elimination method 2. Gauss-Jordan method,

In this article we discuss two numerical iterative methods for solving a system of algebraic equations.

These two methods cannot be applied to any system of equations. It is applicable only when the numerically large coefficients are along the leading / principal diagonal of the coefficient matrix A associated with the system of equations usually represented in the form $AX = B$. Such a system is called a *diagonally dominant system.*

The methods are illustrated for the following system of three independent equations in three unknowns.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system of equations is said to be diagonally dominant if

$$|a_{11}| > |a_{12}| + |a_{13}|, \ |a_{22}| > |a_{21}| + |a_{23}|, \ |a_{33}| > |a_{31}| + |a_{32}|$$

Sometimes we may have to rearrange the given system of equations to meet this requirement. If this condition is satisfied, the solution exists as the iteration process will converge.

5.31 Gauss - Seidel iterative method

We write the system of equations in the form

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \quad \ldots (1)$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \quad \ldots (2)$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \quad \ldots (3)$$

We start with the trial solution (*initial approximation*)

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0.$$

The first approximation are as follows.

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}\cdot 0 - a_{13}\cdot 0\right] = \frac{b_1}{a_{11}}$$

This approximation is immediately used in (2) so that we have

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}\,x_1^{(1)} - a_{23}\cdot 0\right]$$

ie.,
$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}\left(\frac{b_1}{a_{11}}\right)\right]$$

Finally, we use both these approximations in (3), so that we have

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}\,x_1^{(1)} - a_{32}\,x_2^{(1)}\right]$$

This completes first iteration.

The process is continued till we get the solution to the desired degree of accuracy.

WORKED PROBLEMS

26. *Solve the following system of equations by Gauss - Seidel method*

$$10x + y + z = 12$$
$$x + 10y + z = 12$$
$$x + y + 10z = 12$$

>> The given system of equations are diagonally dominant and the equations are put in the form

$$x = \frac{1}{10}\left[12 - y - z\right] \qquad \ldots(1)$$
$$y = \frac{1}{10}\left[12 - x - z\right] \qquad \ldots(2)$$
$$z = \frac{1}{10}\left[12 - x - y\right] \qquad \ldots(3)$$

Let us start with the trial solution $x = 0,\ y = 0,\ z = 0.$

First iteration :

$$x^{(1)} = \frac{1}{10}\left[12 - 0 - 0\right] = 1.2$$
$$y^{(1)} = \frac{1}{10}\left[12 - 1.2 - 0\right] = 1.08$$
$$z^{(1)} = \frac{1}{10}\left[12 - 1.2 - 1.08\right] = 0.972$$

Second iteration :

$$x^{(2)} = \frac{1}{10}\left[12 - 1.08 - 0.972\right] = 0.9948$$

$$y^{(2)} = \frac{1}{10}\left[12 - 0.9948 - 0.972\right] = 1.00332$$

$$z^{(2)} = \frac{1}{10}\left[12 - 0.9948 - 1.00332\right] = 1.000188$$

Third iteration :

$$x^{(3)} = \frac{1}{10}\left[12 - 1.00332 - 1.000188\right] = 0.99965$$

$$y^{(3)} = \frac{1}{10}\left[12 - 0.99965 - 1.000188\right] = 1.00002$$

$$z^{(3)} = \frac{1}{10}\left[12 - 0.99965 - 1.00002\right] = 1.00003$$

Fourth iteration :

$$x^{(4)} = \frac{1}{10}\left[12 - 1.00002 - 1.00003\right] = 0.999995 \approx 1$$

$$y^{(4)} = \frac{1}{10}\left[12 - 1 - 1.00003\right] = 0.999997 \approx 1$$

$$z^{(4)} = \frac{1}{10}\left[12 - 1 - 1\right] = 1$$

Thus $\mathbf{x = 1,\ y = 1,\ z = 1}$

Remark : *The iterative process could have been stopped at the third iteration itself as the solution can be approximated to* $x = 1,\ y = 1,\ z = 1$

---

27. *Solve the following system of equations by Gauss - Seidel method to obtain the final solution correct to three places of decimals*

$$x + y + 54z = 110$$
$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$

>> The given system of equations is not diagonally dominant and hence we have to first rearrange the given system of equations as follows.

$$27x + 6y - z = 85 \quad (|\,27\,| > |\,6\,| + |\,-1\,|)$$

$$6x + 15y + 2z = 72 \quad (|\,15\,| > |\,6\,| + |\,2\,|)$$

$$x + y + 54z = 110 \quad (|\,54\,| > |\,1\,| + |\,1\,|)$$

(*Observe the coefficients column wise for rearrangement*)

These equations are now written as follows.

$$x = \frac{1}{27}\left[85 - 6y + z\right] \quad \ldots(1)$$

$$y = \frac{1}{15}\left[72 - 6x - 2z\right] \quad \ldots(2)$$

$$z = \frac{1}{54}\left[110 - x - y\right] \quad \ldots(3)$$

First iteration : We start with the trial solution $x = 0,\ y = 0,\ z = 0$

$$x^{(1)} = \frac{1}{27}\left[85 - 0 + 0\right] = 3.14815$$

$$y^{(1)} = \frac{1}{15}\left[72 - 6(3.14815) - 0\right] = 3.54074$$

$$z^{(1)} = \frac{1}{54}\left[110 - 3.14815 - 3.54074\right] = 1.91317$$

Second iteration :

$$x^{(2)} = \frac{1}{27}\left[85 - 6(3.54074) + 1.91317\right] = 2.43218$$

$$y^{(2)} = \frac{1}{15}\left[72 - 6(2.43218) - 2(1.91317)\right] = 3.57204$$

$$z^{(2)} = \frac{1}{54}\left[110 - 2.43218 - 3.57204\right] = 1.92585$$

Third iteration :

$$x^{(3)} = \frac{1}{27}\left[85 - 6(3.57204) + 1.92585\right] = 2.42569$$

$$y^{(3)} = \frac{1}{15}\left[72 - 6(2.42569) - 2(1.92585)\right] = 3.57294$$

$$z^{(3)} = \frac{1}{54}\left[110 - 2.42569 - 3.57294\right] = 1.92595$$

Fourth iteration :

$$x^{(4)} = \frac{1}{27}\left[85 - 6(3.57294) + 1.92595\right] = 2.42549$$

$$y^{(4)} = \frac{1}{15}\left[72 - 6(2.42549) - 2(1.92595)\right] = 3.57301$$

$$z^{(4)} = \frac{1}{54}\left[110 - 2.42549 - 3.57301\right] = 1.92595$$

It may be observed that the solution in the third and fourth iteration when approximated to three places of decimals is the same.

Thus $\mathbf{x = 2.426,\ y = 3.573,\ z = 1.926}$

---

28. *Solve the following system of equations by Gauss - Seidel method.*

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

>> The equations are diagonally dominant and hence we first write them in the following form.

$$x = \frac{1}{20}\left[17 - y + 2z\right]$$

$$y = \frac{1}{20}\left[-18 - 3x + z\right]$$

$$z = \frac{1}{20}\left[25 - 2x + 3y\right]$$

We start with the trial solution $x = 0,\ y = 0,\ z = 0$

First iteration :

$$x^{(1)} = \frac{17}{20} = 0.85$$

$$y^{(1)} = \frac{1}{20}\left[-18 - 3(0.85)\right] = -1.0275$$

$$z^{(1)} = \frac{1}{20}\left[25 - 2(0.85) + 3(-1.0275)\right] = 1.0109$$

Second iteration :

$$x^{(2)} = \frac{1}{20}\left[17 - (-1.0275) + 2(1.0109)\right] = 1.0025$$

$$y^{(2)} = \frac{1}{20}\left[-18 - 3(1.0025) + 1.0109\right] = -0.9998$$

$$z^{(2)} = \frac{1}{20}\left[(25 - 2(1.0025) + 3(-0.9998)\right] = 0.9998$$

Third iteration :

$$x^{(3)} = \frac{1}{20}\left[17-(-0.9998)+2(0.9998)\right] = 0.99997$$

$$y^{(3)} = \frac{1}{20}\left[-18-3(0.99997)+0.9998\right] = -1.0000055$$

$$z^{(3)} = \frac{1}{20}\left[(25-2(0.99997)+3(-1.0000055)\right] = 1.0000022$$

Thus $x = 1$, $y = -1$, $z = 1$ is the required solution.

---

29. *Employ Gauss - Seidel iteration method to solve*

$$5x+2y+z = 12$$

$$x+4y+2z = 15$$

$$x+2y+5z = 20$$

Carryout 4 iterations taking the initial approximation to the solution as $(1, 0, 3)$

>> The given system of equations are diagonally dominant and we put them in the following form.

$$x = \frac{1}{5}[12-2y-z]$$

$$y = \frac{1}{4}[15-x-2z]$$

$$z = \frac{1}{5}[20-x-2y]$$

By data, $x^{(0)} = 1$, $y^{(0)} = 0$, $z^{(0)} = 3$

First iteration :

$$x^{(1)} = \frac{1}{5}[12-2(0)-3] = 1.8$$

$$y^{(1)} = \frac{1}{4}[15-1.8-2(3)] = 1.8$$

$$z^{(1)} = \frac{1}{5}[20-1.8-2(1.8)] = 2.92$$

Second iteration :

$$x^{(2)} = \frac{1}{5}[12-2(1.8)-2.92] = 1.096$$

$$y^{(2)} = \frac{1}{4}[15-1.096-2(2.92)] = 2.016$$

$$z^{(2)} = \frac{1}{5}[20-1.096-2(2.016)] = 2.9744$$

Third iteration :

$$x^{(3)} = \frac{1}{5}\,[\,12 - 2\,(\,2.016\,) - 2.9744\,] \quad = 0.99872$$

$$y^{(3)} = \frac{1}{4}\,[\,15 - 0.99872 - 2\,(\,2.9744\,)\,] \quad = 2.01312$$

$$z^{(3)} = \frac{1}{5}\,[\,20 - 0.99872 - 2\,(\,2.01312\,)\,] = 2.995$$

Fourth iteration :

$$x^{(4)} = \frac{1}{5}\,[\,12 - 2\,(\,2.01312\,) - 2.995\,] \quad = 0.995752$$

$$y^{(4)} = \frac{1}{4}\,[\,15 - 0.995752 - 2\,(\,2.995\,)\,] \quad = 2.003562$$

$$z^{(4)} = \frac{1}{5}\,[\,20 - 0.995752 - 2\,(\,2.003562\,)\,] = 2.9994248$$

Thus the solution after four iterations correct to four decimal places is given by

$\mathbf{x = 0.9958\,,\ \ y = 2.0036\,,\ \ z = 2.9994}$

---

30. *Sove the sytem of equations represented $AX = B$ by applying Gauss - seidel iterative method where*

$$A = \begin{bmatrix} 28 & 4 & -1 \\ 1 & 3 & 10 \\ 2 & 17 & 4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 32 \\ 24 \\ 35 \end{bmatrix}$$

Perform five iterations to obtain the solution correct to four decimal places.

>> It can be seen that numerically large coefficients are not in the principal diagonal. We need to interchange the second and third row in A so that we have the following system of equations in the diagonally dominant form.

$$28x + 4y - 1z = 32$$

$$2x + 17y + 4z = 35$$

$$x + 3y + 10z = 24$$

From these we have,

$$x = \frac{1}{28}\,[\,32 - 4y + z\,]$$

$$y = \frac{1}{17}\,[\,35 - 2x - 4z\,]$$

$$z = \frac{1}{10}\,[\,24 - x - 3y\,]$$

We start with the trial solution $x = 0,\ \ y = 0,\ \ z = 0.$

First iteration :

$$x^{(1)} = \frac{32}{28} = 1.1429$$

$$y^{(1)} = \frac{1}{10}\,[\,35 - 2\,(\,1.1429\,)\,] = 1.9244$$

$$z^{(1)} = \frac{1}{10}\left[\,24 - 1.1429 - 3\,(\,1.9244\,)\,\right] = 1.7084$$

Proceeding on the same lines the values obtained at various iterative levels are tabulated.

	Second iteration	Third iteration	Fourth iteration	**Fifth iteration**
x	0.929	0.9876	0.9933	**0.9936**
y	1.5476	1.509	1.507	**1.507**
z	1.8428	1.8485	1.8486	**1.8485**

Thus the solution correct to four decimal palces is given by

$$\mathbf{x = 0.9936,\; y = 1.5070,\; z = 1.8485}$$

5.32 Relaxation method

The method is illustrated for the following diagonally dominant system of three independent equations in three unknowns.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

From these equations the *residuals* R_1 , R_2 , R_3 are defined as follows.

$$R_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - b_1$$

$$R_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 - b_2$$

$$R_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 - b_3$$

The values of x_1 , x_2 , x_3 that make the residuals R_1 , R_2 , R_3 simultaneously zero constitutes the *exact solution* of the system of equations. If this is not possible we need to make the residuals as close to zero as possible. (*residuals relaxed to zero*)

The values of x_1 , x_2 , x_3 at that stage constitutes an *approximatate solution.*

The relaxation method aims in reducing the values of residuals as close to zero as possible by modifying the value of the variables at every stage.

At every step the numerically largest residual is reduced to zero / almost zero by choosing an appropriate integral value for the corresponding variable.

These integral values are called the increments in x_1, x_2, x_3 denoted by $\Delta x_1, \Delta x_2, \Delta x_3$ respectively. If the system possess exact soultion then R_1, R_2, R_3 becomes zero simultaneously and the required solution will be the sum of all these increments.

That is, $x_1 = \sum \Delta x_1, \quad x_2 = \sum \Delta x_2, \quad x_3 = \sum \Delta x_3$

When the system doesnot possess exact solution, we need to *continue the process* by magnifying the prevailing residuals *on multiplication with* 10. Later we divide by 10 to obtain the solution correct to one decimal place.

The process (*multiplication with* 10) will be continued to meet the requirement of the solution to the desired accuracy. (*decimal places*)

Remarks :

1. *The salient feature of this method is that we can obtain the solution of a diagonally dominant system of equations to the desired accuracy without the extensive use of calculator.*

2. *First three problems in this method are solved with explanation in the Remarks column to get a clear insight of the method.*

WORKED PROBLEMS

31. *Solve the following system of equations by relaxation method :*

$$10x + y + z = 12$$

$$x + 10y + z = 12$$

$$x + y + 10z = 12$$

>> The given system of equations are diagonally dominant

Let $R_1 = 10x + y + z - 12$

$R_2 = x + 10y + z - 12$

$R_3 = x + y + 10z - 12$, be the residuals.

The solution for the system is presented in the following table called the *relaxation table*. $\Delta_x, \Delta_y, \Delta_z$ represent the increments for x, y, z respectively.

Δ_x	Δ_y	Δ_z	R_1	R_2	R_3	Remarks
0	0	0	[redacted]	-12	-12	Initially no increment is given. Values of residuals are written. A numerically large residual $R_1 = -12$ is made close to zero by giving $\Delta_x = 1$. That is $(x, y, z) = (1, 0, 0)$ being substituted in the 3 equations.
1	0	0	-2	[redacted]	-11	
0	1	0	-1 $(1-2)$	-1 $(10-11)$	[redacted] $(1-11)$	Agian a numerically large residual $R_2 = -11$ is made close to zero giving $\Delta_y = 1$. We put $y = 1$ in the 3 equations & the **resultant is to be added** to the prevailing $R_1 = -2$, $R_2 = -11$, $R_3 = -11$. **Note** : This is equivalent to substituting $(x, y, z) = (1, 1, 0)$ in the 3 equations to find R_1, R_2, R_3
0	0	1	[redacted]	[redacted]	[redacted]	The numerically largest residual $R_3 = -10$ becomes exactly zero for $\Delta_z = 1$. We put $z = 1$ in the three equations and the **resultant is added** to $R_1 = -1$, $R_2 = -1$, $R_3 = -10$. Along with R_3, the other residuals too become zero.

It can be seen that R_1, R_2, R_3 are all zero and the exact solution of the given system is $x = \sum \Delta_x = 1+0+0 = 1$; $y = \sum \Delta_y = 0+1+0 = 1$, $z = \sum \Delta_z = 0+0+1 = 1$

Thus $(x, y, z) = (1, 1, 1)$ is the solution of the given system of equations.

Note : *This problem has been solved by Gauss - Seidel mehtod* [Problem - 26]

---

32. *Solve the following system of equations by relaxation method.*

$$12x_1 + x_2 + x_3 = 31$$

$$2x_1 + 8x_2 - x_3 = 24$$

$$3x_1 + 4x_2 + 10x_3 = 58$$

>> The given system of equations are diagonally dominant.

Let $R_1 = 12x_1 + x_2 + x_3 - 31$

$R_2 = 2x_1 + 8x_2 - x_3 - 24$

$R_3 = 3x_1 + 4x_2 + 10x_3 - 58$, be the residuals.

Let Δx_1, Δx_2, Δx_3 respectively represent the increments for x_1, x_2, x_3 in the following relaxation table.

Δx_1	Δx_2	Δx_3	R_1	R_2	R_3	**Remarks**
0	0	0	-31	-24	[redacted]	No increment given
0	0	6	-25 $(6-31)$	[redacted] $(-6-24)$	2 $(60-58)$	$R_3 = -58$ is numerically large and $x_3 = 6$ gives $10 \times 6 - 58 = 2$ is close to 0.
0	4	0	[redacted] $(4-25)$	2 $(32-30)$	18 $(16+2)$	$R_2 = -30$ is numerically large and $x_2 = 4$ gives $8 \times 4 - 30 = 2$ is close to zero.
2	0	0	3 $(24-21)$	6 $(4+2)$	[redacted] $(6+18)$	$R_1 = -21$ is numercially large and $x_1 = 2$ gives $12 \times 2 - 21 = 3$ is close to zero.
0	0	-2	1 $(-2+3)$	8 $[-(-2)+6]$	4 $(-20+24)$	$R_3 = 24$ is large and $x_3 = -2$ gives $10 \times -2 + 24 = 4$ is close to zero.
0	-1	0	[redacted] $(-1+1)$	[redacted] $(-8+8)$	[redacted] $(-4+4)$	$R_2 = 8$ is large and $x_2 = -1$ gives $-8+8 = 0$. Other residuals too become zero.

R_1, R_2, R_3 are all zero and the exact solution of the given system is

$$x_1 = \sum \Delta x_1 = 2\,;\; x_2 = \sum \Delta x_2 = 4 - 1 = 3\,;\; x_3 = \sum \Delta x_3 = 6 - 2 = 4$$

Thus $(x_1, x_2, x_3) = (2, 3, 4)$ is the solution of the given system of equations.

---

33. *Obtain the solution of the following system of equations correct to three decimal places by applying relaxation method.*

$$x + y + 54z = 110$$
$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$

>> The given system is not diagonally dominant and we need to rearrange the system by observing the coefficients column wise.

The diagonally dominant system in the residual form is as follows.

$$R_1 = 27x + 6y - z - 85$$
$$R_2 = 6x + 15y + 2z - 72$$
$$R_3 = x + y + 54z - 110$$

Let Δ_x, Δ_y, Δ_z respectively represent the increments for x, y, z in the following relaxation table.

Δ_x	Δ_y	Δ_z	R_1	R_2	R_3	**Remarks**
0	0	0	– 85	– 72	[redacted]	Numerically large residuals are made close to zero.
0	0	2	[redacted]	– 68	– 2	
3	0	0	– 6	[redacted]	1	
0	3	0	12	– 5	4	It is not possible to minimize 12 further since the coefficient of x is 27.
($\sum$) 3	3	2	12	– 5	4	
30	30	20	[redacted]	– 50	40	We multiply the entire last row by 10 and continue with the same process.
– 4	0	0	12	[redacted]	36	* It is not possible to further minimize. We add all the increments and multiply the entire row by 10.
0	5	0	[redacted]	1	41	
– 2	0	0	– 12	– 11	[redacted]	
0	0	– 1	– 11	– 13	– 15 *	
($\sum$) 24	35	19	– 11	– 13	– 15	only the increments are added
240	350	190	– 110	– 130	[redacted]	Entire preceeding row is multiplied by 10. Process is continued.
0	0	3	– 113	[redacted]	12	* It is not possible to further minimize.
0	8	0	[redacted]	– 4	20	
2	0	0	– 11	8	22 *	
($\sum$) 242	358	193	– 11	8	22	only the increments are added
2420	3580	1930	– 110	80	[redacted]	
0	0	– 4	[redacted]	72	4	
4	0	0	2	[redacted]	8	Entire preceeding row is multiplied by 10. Process is continued.
0	– 6	0	[redacted]	6	2	
1	0	0	– 7	12	3	
($\sum$) 2425	3574	1926	– 7	12	3	only the increments are added.

It should be noted that we have multiplied by 10 three times. Hence we have

$$10^3 x = 2425, \quad 10^3 y = 3574, \quad 10^3 z = 1926 \quad \text{and correspondingly}$$

$$10^3 R_1 = -7, \quad 10^3 R_2 = 12, \quad 10^3 R_3 = 3$$

Therefore it is equivalent to say that

$$x = 2.425, \quad y = 3.574, \quad z = 1.926$$

has reduced, (*relaxed*) the residuals R_1, R_2, R_3 to – 0.007, 0.012, 0.003 close to zero.

Thus the solution of the given system of equations correct to three decimal places is

$$x = 2.425, \quad y = 3.574, \quad z = 1.926$$

Remark : *Refer to problem-27, where we have solved the prblem by Gauss-Seidel method.*

---

34. *Solve the following system of equations by relaxation method obtaining the solution correct to two decimal places.*

$$9x - 2y + z = 50$$
$$x + 5y - 3z = 18$$
$$-2x + 2y + 7z = 19$$

>> The system is diagonally dominant.

Let $R_1 = 9x - 2y + z - 50$

$R_2 = x + 5y - 3z - 18$

$R_3 = -2x + 2y + 7z - 19,$ be the residuals.

Also let Δ_x, Δ_y, Δ_z represent the increments for x, y, z respectively in the following relaxation table.

Δ_x	Δ_y	Δ_z	R_1	R_2	R_3	Remarks
0	0	0		−18	−19	
6	0	0	[redacted]	−12		
0	0	4	8		[redacted]	
0	5	0	−2	[redacted]	7	
60	50	40	−20	10		Increments are added and the entire row is multiplied by 10.
0	0	−10	−30		[redacted]	
0	−8	0	−14	[redacted]		
0	0	2		−6	[redacted]	
1	0	0	[redacted]	−5	−4	
610	420	320	−30		−40	Increments are added and the entire row is multiplied by 10
0	10	0		[redacted]	−20	
6	0	0	[redacted]	6		
0	0	5		−9	[redacted]	
−1	0	0	[redacted]		5	
0	2	0	−4	[redacted]	9	
615	432	325	−4	0	9	Increments are being added

Hence we have $10^2 x = 615$, $10^2 y = 432$, $10^2 z = 325$ and correspondingly

$$10^2 R_1 = -4, \quad 10^2 R_2 = 0, \quad 10^2 R_3 = 9$$

$\therefore$ $x = 6.15$, $y = 4.32$, $z = 3.25$ has reduced (*relaxed*) residuals close to zero.

Thus the required **solution correct to two decimal places** is given by

$$\mathbf{x = 6.15, \quad y = 4.32, \quad z = 3.25}$$

---

35. *Obtain the solution of the following system of equations correct to one decimal place by applying relaxation method.*

$$10a - 2b + c = 12$$
$$a + 9b - c = 10$$
$$2a - b + 11c = 20$$

>> The given system is diagonally dominant.

Let $R_1 = 10a - 2b + c - 12$

$R_2 = a + 9b - c - 10$

$R_3 = 2a - b + 11c - 20$, be the residuals.

Also let Δ_a, Δ_b, Δ_c respectively represent the increments for a, b, c in the following relaxation table.

Δ_a	Δ_b	Δ_c	R_1	R_2	R_3	**Remarks**
0	0	0	−12	−10	[redacted]	Numerically large residuals are made close to zero.
0	0	2	−10	[redacted]	2	
0	1	0	[redacted]	−3	1	
1	0	0	−2	−2	3	
10	10	20	−20	−20	[redacted]	Increments are added and the entire row is multiplied by 10
0	0	−3	[redacted]	−17	−3	
2	0	0	−3	[redacted]	1	
0	2	0	[redacted]	3	−1	
1	0	0	3	4	1	
13	12	17	3	4	1	Increments are added

Hence we have, $10\,a = 13, 10\,b = 12, 10\,c = 17$ and correspondingly

$$10\,R_1 = 3, 10\,R_2 = 4, 10\,R_3 = 1$$

Therefore $a = 1.3,\ b = 1.2,\ c = 1.7$ has reduced (*relaxed*) the residuals

R_1, R_2, R_3 respectively to $0.3, 0.4, 0.1$ close to zero.

Thus $\mathbf{a = 1.3, b = 1.2, c = 1.7}$ is the required solution correct to one decimal place.

---

36. *Solve by relaxation method :*

$$10x + 2y + z = 9$$

$$x + 10y - z = -22$$

$$-2x + 3y + 10z = 22$$

>> The given system is diagonally dominant.

Let $R_1 = 10x + 2y + z - 9$

$R_2 = x + 10y - z + 22$

$R_3 = -2x + 3y + 10z - 22$, be the residuals.

Also let $\Delta_x, \Delta_y, \Delta_z$ respectively represent the increments for x, y, z in the following relaxation table.

Δ_x	Δ_y	Δ_z	R_1	R_2	R_3	**Remark**
0	0	0	-9	$+22$	[redacted]	
0	0	2	-7	20	-2	Numerically large residuals are made close to zero.
0	-2	0	[redacted]	0	-8	
1	0	0	-1	1	[redacted]	
0	0	1	[redacted]	[redacted]	[redacted]	All the residuals are zero.

The system has exact solution given by

$$x = \sum \Delta_x = 1, y = \sum \Delta_y = -2, z = \sum \Delta_z = 3$$

Thus $(x, y, z) = (1, -2, 3)$ **is the required solution.**

---

37. *Solve by relaxation method :*

$$10x - 2y - 3z = 205$$
$$-2x + 10y - 2z = 154$$
$$-2x - y + 10z = 120$$

>> The given system is diagonally dominant.

Let $R_1 = 10x - 2y - 3z - 205$

$R_2 = -2x + 10y - 2z - 154$

$R_3 = -2x - y + 10z - 120$, be the residuals.

Also let Δ_x, Δ_y, Δ_z respectively represent the increments for x, y, z in the following relaxation table.

Δ_x	Δ_y	Δ_z	R_1	R_2	R_3	**Remark**
0	0	0	[redacted]	– 154	– 120	
21	0	0	+ 5	[redacted]	– 162	
0	20	0	– 35	4	[redacted]	
0	0	18	[redacted]	– 32	– 2	Numerically large residuals are made
9	0	0	1	[redacted]	– 20	close to zero.
0	5	0	– 9	0	[redacted]	
0	0	3	[redacted]	– 6	5	
2	0	0	2	[redacted]	1	
0	1	0	[redacted]	[redacted]	[redacted]	All the residuals are zero.

The system of equations has exact solution given by

$$x = \sum \Delta_x = 32 \; ; \; y = \sum \Delta_y = 26 \; ; \; z = \sum \Delta_z = 21$$

Thus $(x, y, z) = (32, 26, 21)$ **is the required solution.**

5.4 Rayleigh's power method

Preamble : We recall the definition of eigen values and the corresponding eigen vectors of a square matrix. [*Vol*-1, *Unit*-VIII].

Given a square matrix A, if there exists a scalar λ and a non zero column matrix X, such that $AX = \lambda X$, then λ is called an eigen value of A and X is called an eigen vector of A corresponding to an eigen value λ.

We have also discussed the theoretical method of finding all the eigen values and the corresponding eigen vectors of A.

Rayleigh's power method is an iterative method to determine the numerically largest eigen value (*dominant eigen value*) and the corresponding eigen vector of a square matrix. The method is well suited for computers.

Working procedure for problems

- Suppose A is the given square matrix, we assume initially an eigen vector (*column matrix*) X_0 in a simple form like $[1, 0, 0]'$ or $[0, 1, 0]'$ or $[0, 0, 1]'$ or $[1, 1, 1]'$ and find the matrix product AX_0 which will also be a column matrix.
- We take out the largest element as the common factor (*this technique is called normalization*) to obtain $A X_0 = \lambda^{(1)} X^{(1)}$.
- We then compute $A X^{(1)}$ and again put it in the form $A X^{(1)} = \lambda^{(2)} X^{(2)}$ by normalization.
- This iterative process is continued till two consecutive iterative values of λ and X are same upto a desired degree of accuracy.
- The values so obtained are respectively the largest eigen value and the corresponding eigen vector of the given square matrix A.

WORKED PROBLEMS

38. *Find the largest eigen value and the corresponding eigen vector of the matrix A by the power method given that*

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

>> Let $X^{(0)} = [1, 0, 0]'$ be the initial eigen vector.

$$\therefore \quad AX^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 0 \\ 2.86 \end{bmatrix} = 2.93 \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 0 \\ 2.96 \end{bmatrix} = 2.98 \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 0 \\ 2.98 \end{bmatrix} = 2.99 \begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \begin{bmatrix} 2.997 \\ 0 \\ 2.994 \end{bmatrix} = 2.997 \begin{bmatrix} 1 \\ 0 \\ 0.999 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

Thus the **largest eigen value** is approximately **3** and the corresponding **eigen vector is [1, 0, 1]′.**

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39. *Find the dominant eigen value and the corresponding eigen vector of the matrix*

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

by power method taking the initial eigen vector as [1, 1, 1]′ .

>> By data $X^{(0)} = [\,1,\ 1,\ 1\,]'$ is the initial eigen vector.

$$AX^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 7.34 \\ -2.67 \\ 4.01 \end{bmatrix} = 7.34 \begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 7.82 \\ -3.63 \\ 4.01 \end{bmatrix} = 7.82 \begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \begin{bmatrix} 7.94 \\ -3.89 \\ 3.99 \end{bmatrix} = 7.94 \begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 7.98 \\ -3.97 \\ 3.99 \end{bmatrix} = 7.98 \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

Thus the dominant eigen value is **8** and the corresponding eigen vector is
[1, − 0.5, 0.5] or **[2 , − 1 , 1] equivalently.**

---

40. *Using Rayleigh's power method find numerically the largest eigen value and the corresponding eigen vector of the matrix*

$$A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

>> Let $X^{(0)} = [\,1, \ 0, \ 0\,]'$ be the initial eigen vector.

$$AX^{(0)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25 \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2 \begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.12 \\ 1.72 \end{bmatrix} = 25.18 \begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

We observe that $X^{(2)} = X^{(3)}$

Thus the numerically largest eigen value of A is **25.18** and the corresponding eigen vector is **[1, 0.04, 0.07] ′**

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41. *Find the numerically largest eigen value and the corresponding eigen vector of the matrix* $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ *by taking the initial approximation to the eigen vector as* $[1, 0.8, -0.8]^T$. *Perform 5 iterations.*

>> By data $X_0 = [1, 0.8, -0.8]^T$ is the initial eigen vector. Hence we have,

$$AX^{(0)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix} = 5.6\begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \begin{bmatrix} 5.86 \\ 5.72 \\ -5.72 \end{bmatrix} = 5.86\begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \begin{bmatrix} 5.96 \\ 5.92 \\ -5.92 \end{bmatrix} = 5.96\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 5.96 \\ -5.96 \end{bmatrix} = 5.98\begin{bmatrix} 1 \\ 0.997 \\ -0.997 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.997 \\ -0.997 \end{bmatrix} = \begin{bmatrix} 5.994 \\ 5.988 \\ -5.988 \end{bmatrix} = 5.994\begin{bmatrix} 1 \\ 0.999 \\ -0.999 \end{bmatrix}$$

Thus after 5 iterations the numerically largest eigen value is **5.994** and the corresponding eigen vector is $[\mathbf{1}, \mathbf{0.999}, \mathbf{-0.999}]^T$

---

42. *Find the largest eigen value and the corresponding eigen vector of the matrix A, by using the power method by taking initial vector as* $[1, 1, 1]^T$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

>> By data $X^{(0)} = [1, 1, 1]^T$. Hence we have

$$AX^{(0)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = 4\begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -3.5 \\ 2.5 \end{bmatrix} = 3.5\begin{bmatrix} 0.71 \\ -1 \\ 0.71 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.71 \\ -1 \\ 0.71 \end{bmatrix} = \begin{bmatrix} 2.42 \\ -3.42 \\ 2.42 \end{bmatrix} = 3.42\begin{bmatrix} 0.708 \\ -1 \\ 0.708 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.708 \\ -1 \\ 0.708 \end{bmatrix} = \begin{bmatrix} 2.416 \\ -3.416 \\ 2.416 \end{bmatrix} = 3.416\begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix} = \begin{bmatrix} 2.4146 \\ -3.4146 \\ 2.4146 \end{bmatrix} = 3.4146\begin{bmatrix} 0.7071 \\ -1 \\ 0.7071 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

Hence the largest **eigen value is 3.4146** and the corresponding **eigen vector is**
$[0.7071, -1, 0.7071]^T$

43. *Use the power method to find the dominant eigen value and the corresponding eigen vector of the matrix* $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ *with the initial eigen vector* $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. *Carry out 5 iterations.*

>> It may be observed that $X^{(0)}$ given here is $X^{(2)}$ in problem-42 which we got starting with $[1, 1, 1]^T$ as the initial vector $X^{(0)}$. The answer is same as of problem-42.

EXERCISES

Solve the following system of equations by ***(a) Gauss-Seidel iterative method (b) Relaxation method*** *(Obtain the solution correct to three decimal places)*

1. $5x + 2y + z = 12, \ x + 4y + 2z = 15, \ x + 2y + 5z = 0$

2. $28x + 4y - z = 32, \ 2x + 17y + 4z = 35, \ x + 3y + 10z = 24$

3. $10x - 2y + z = 12, x + 9y - z = 10, \ 2x - y + 11z = 20$

4. $x + 17y - 2z = 48, \ 2x + 2y + 18z = 30, \ 30x - 2y + 3z = 48$

5. $9x - y + 2z = 9, \ x + 10y - 2z = 15, \ 2x - 2y - 13z = -17$

Obtain the solution of the following system of equations by relaxation method

6. $5x - y - z = 3, \ -x + 10y - 2z = 7, \ -x - y + 10z = 8$

7. $10x - 2y - 3z + 205 = 0, \ -2x + 10y - 2z + 154 = 0 \ -2x - y + 10z + 120 = 0$

Use rayleigh's power method to find the largest eigen value and the corresponding eigen vector of the following matrices.

8. $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (Take $X_0 = [1, 0, 0]'$)

9. $\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ (Take $X_0 = [0, 1, 0]'$)

10. $\begin{bmatrix} 10 & 2 & 1 \\ 2 & 10 & 1 \\ 2 & 1 & 10 \end{bmatrix}$ (Take $X_0 = [0, 0, 1]'$)

ANSWERS

1. 1 , 4.5 , − 2

2. 0.994 , 1.507 , 1.849

3. 1.262 , 1.159 , 1.694

4. 1.675 , 2.862 , 1.163

5. 0.917 , 1.647 , 1.195

6. 1 , 1 , 1

7. − 32 , − 26 , − 21

8. $\lambda = 4$; $[1, 0.5, 0]'$

9. $\lambda = 6$; $[1, 0, 1]'$

10. $\lambda = 13$; $[1, 1, 1]'$